# Advanced ODE-Lecture 7
# Linear Boundary Value Problem


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# Outline

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# Motivation

- **Linear BVP can be regarded as an extension of the linear IVP although they are quite different from methods, results for existence, uniqueness and solution formulae.**

- **We concern its existence and uniqueness of solution for linear BVP, and how to solve the linear BVP in general.**

# Existence and Uniqueness of Linear BVP

The general linear boundary value problem is given by:

$$x' = A(t)x + f(t), \quad Cx(a) + Dx(b) = \eta, \qquad \text{(BVP)}$$

where $A(t)$, $C$ and $D$ is $n \times n$ matrices and $x$, $f(t)$ and $\eta$ are $n$ dimensional vectors, $A(t)$ and $f(t)$ are both in $C[a,b]$. In particular, if $C = I$ and $D = O$, the linear BVP is a well-treated IVP.

**Theorem 7.1** Let $\Phi(t)$ be a fundamental matrix of $x' = A(t)x$, then the following statements are equivalent:

(1) The linear homogeneous BVP (HBVP) $x' = A(t)x$, $Cx(a) + Dx(b) = 0$ has only zero solution;

(2) $n$ dimensional matrix $B = C\Phi(a) + D\Phi(b)$ is nonsingular;

(3) The linear BVP has a unique solution.

**Proof: (1) $\Leftrightarrow$ (2):** Since $\Phi(t)$ is a fundamental matrix, the general solution of

$$x' = A(t)x \quad \text{is} \quad x = \Phi(t)c,$$

where $c$ is an arbitrary $n$ dimensional constant vector. It is easy to see that

$$[C\Phi(a) + D\Phi(b)]c = 0,$$

which has only solution of $c = 0$ $\Leftrightarrow$ the coefficient matrix $B = C\Phi(a) + D\Phi(b)$ is nonsingular.

**(2) $\Leftrightarrow$ (3):** Let $\varphi(t)$ be a solution of

$$x' = A(t)x + f(t).$$

Then, its general solution is $x = \Phi(t)c + \varphi(t)$, where $c$ is an arbitrary $n$ dimensional constant vector. Substituting the general solution into the linear BVP, we have

$$Bc = \eta - C\varphi(a) - D\varphi(b).$$

From the above, we obtain immediately that the linear algebraic system has a unique solution $\Leftrightarrow$ its coefficient matrix $B$ is nonsingular. This concludes the result. □

# Green Function and Solution Formula

**Remark 7.1** Theorem 7.1 gives the characterization of the existence and uniqueness of solution to the linear BVP. If $B = C\Phi(a) + D\Phi(b)$ is nonsingular, or the linear HBVP $x' = A(t)x$, $Cx(a) + Dx(b) = 0$ has unique solution (zero solution), then, the linear BVP has a unique solution.

**Remark 7.2** If $\Phi(t)$ is found, $B = C\Phi(a) + D\Phi(b)$ can be computed.

How to solve the linear BVP? What is its solution structure?

First, we need to define a Green matrix function and compute the Green function.

Second, we can solve the linear BVP with obtained Green matrix function based on Theorem 7.1.

**Definition 7.1** $n$ dimensional matrix function $G(t,s)$ is called Green matrix function of the linear HBVP if it satisfies the properties:

(1) For any $s \in (a,b)$, $\frac{\partial G}{\partial t}(t,s) = A(t)G(t,s)$ when $t \in [a,b]$ and $t \neq s$;

(2) For any $s \in (a,b)$, $CG(a,s) + DG(b,s) = 0$, $s \in (a,b)$;

(3) $G(t,s)$ is continuous on $a \le t \le b$, $a \le s \le b$, $t \neq s$, and has a jump discontinuous at $t = s$ such that

$$G(s+0,s) - G(s-0,s) = I, \quad s \in (a,b).$$

**Theorem 7.2** Suppose that $A(t)$ is continuous on $[a,b]$, and the linear HBVP has only zero solution. Then, there exists uniquely a Green matrix function $G(t,s)$, which is defined by (G-1), such that the linear BVP has a unique solution

$$x(t) = \Phi(t)B^{-1}\eta + \int_a^b G(t,s) f(s)ds, \qquad \text{(G-0)}$$

where $\Phi(t)$ is a fundamental matrix of $x' = A(t)x$ and $B = C\Phi(a) + D\Phi(b)$.

**Proof**: Let

$$G(t,s)=\begin{cases}-\Phi(t)B^{-1}D\Phi(b)\Phi^{-1}(s), & a\le t<s,\\ \Phi(t)B^{-1}C\Phi(a)\Phi^{-1}(s), & s<t\le b.\end{cases} \tag{G-1}$$

We verify three conditions by the definition of a Green matrix function. Clearly, the conditions (1) and (3) are satisfied by the definition of $G(t,s)$ in (G-1). Now we verify the condition (2). Since

$$\begin{aligned} CG(a,s)+DG(b,s) &= -C\Phi(a)B^{-1}D\Phi(b)\Phi^{-1}(s)+D\Phi(b)B^{-1}C\Phi(a)\Phi^{-1}(s)\\ &= -C\Phi(a)B^{-1}D\Phi(b)\Phi^{-1}(s)-D\Phi(b)B^{-1}D\Phi(b)\Phi^{-1}(s)\\ &\quad +D\Phi(b)B^{-1}D\Phi(b)\Phi^{-1}(s)+D\Phi(b)B^{-1}C\Phi(a)\Phi^{-1}(s)\\ &= -[C\Phi(a)+D\Phi(b)]B^{-1}D\Phi(b)\Phi^{-1}(s)\\ &\quad +D\Phi(b)B^{-1}[D\Phi(b)+C\Phi(a)]\Phi^{-1}(s)\\ &= -BB^{-1}D\Phi(b)\Phi^{-1}(s)+BB^{-1}D\Phi(b)\Phi^{-1}(s)=0, \end{aligned}$$

then, the defined $G(t,s)$ is a Green matrix function of the linear HBVP by definition.

As for the uniqueness of $G(t,s)$, we show by contradiction. If there were two Green matrix functions $G_1(t,s)$ and $G_2(t,s)$ such that $G_1(t,s) \neq G_2(t,s)$ for $(t,s) \in [a,b] \times [a,b]$, $t \neq s$;

Let $G(t,s) = G_1(t,s) - G_2(t,s) \neq 0$ for all $(t,s) \in [a,b] \times [a,b]$, $t \neq s$.

First, for $\forall s \in [a,b]$, $G(t,s)$ is a continuous function on $t \in [a,b]$ and $t \neq s$; when $t = s$, in view of (3) in the definition of the Green matrix function, we have

$$G(s+0,s) - G(s-0,s) = G_1(s+0,s) - G_2(s+0,s) - G_1(s-0,s) + G_2(s-0,s)$$

$$= I - I = 0.$$

Then, after we define $G(s,s) = G(s+0,s) = G(s-0,s)$, we obtain that for $\forall s \in [a,b]$, $G(t,s)$ is continuous on $t \in [a,b]$.

Second, since $A(t)$ is continuous and

$$\frac{\partial G}{\partial t}(t,s) = A(t)G(t,s), \quad t \neq s,$$

we obtain that for $a \le s < t$ or $b \ge s > t$, $G(t,s)$ is a solution of $x' = A(t)x$, so as to be the solution of the linear HBVP except for the point at $t \neq s$, which implies $G(t,s) \equiv 0$, $t \neq s$. This is a contradiction by assumption. The uniqueness of the Green matrix function is proved.

Finally, we show that (G-1) is a solution of the linear BVP. We submit (G-1) into (G-0) and have

$$x(t) = \Phi(t)B^{-1}\eta + \int_a^t \Phi(t)B^{-1}C\Phi(a)\Phi^{-1}(s)f(s)ds - \int_t^b \Phi(t)B^{-1}D\Phi(b)\Phi^{-1}(s)f(s)ds.$$

Taking derivative above on both sides, we have

$$x'(t) = A(t)\Phi(t)B^{-1}\eta + A(t)\int_a^t \Phi(t)B^{-1}C\Phi(a)\Phi^{-1}(s)f(s)ds + \Phi(t)B^{-1}C\Phi(a)\Phi^{-1}(t)f(t)$$

$$- A(t)\int_t^b \Phi(t)B^{-1}D\Phi(b)\Phi^{-1}(s)f(s)ds + \Phi(t)B^{-1}D\Phi(b)\Phi^{-1}(t)f(t)$$

$$= A(t)x(t) + \Phi(t)B^{-1}B\Phi^{-1}(t)f(t) = A(t)x(t) + f(t);$$

which implies that (G-0) is a solution of $x' = A(t)x + f(t)$. Moreover, we show that (G-0) satisfies $Cx(a) + Dx(b) = \eta$. By the condition (2) of the Green matrix function, we have

$$Cx(a) + Dx(b) = C\Phi(a)B^{-1}\eta + \int_a^b CG(a,s)f(s)ds + D\Phi(b)B^{-1}\eta + \int_a^b DG(b,s)f(s)ds$$

$$= \eta + \int_a^b [CG(a,s) + DG(b,s)]f(s)ds = \eta.$$

Therefore, (G-0) is a unique solution of the linear BVP. This is the end of proof. □

**Remark 7.3** Theorem 7.2 gives a solution formula based on $\Phi(t)$.

**Example 7.1** Find the solution of the following BVP:

$$\begin{cases} x_1' = x_2, & x_1(0) = 0, \\ x_2' = -x_1 + 1, & x_2(\pi) = 0. \end{cases}$$

**Solution:** We write the BVP into the standard form and then we have

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad f(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \eta = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We may find the fundamental matrix solution of $x' = Ax$ as follows:

$$\Phi(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

Hence,

$$B = C\Phi(0) + D\Phi(\pi) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = B^{-1},$$

which is nonsingular. Therefore, the problem of interests has a unique solution.

We compute $G(t,s)$ with the formula (G-1) as follows:

$$G(t,s)=\begin{cases}-\Phi(t)B^{-1}D\Phi(\pi)\Phi^{-1}(s), & 0\le t<s,\\ \Phi(t)B^{-1}C\Phi(0)\Phi^{-1}(s), & s<t\le\pi,\end{cases}$$

$$=\begin{cases}-\begin{pmatrix}\sin t\sin s & \sin t\cos s\\ \cos t\sin s & \cos t\cos s\end{pmatrix}, & 0\le t<s,\\ \begin{pmatrix}\cos t\cos s & -\cos t\sin s\\ -\sin t\cos s & \sin t\sin s\end{pmatrix}, & s<t\le\pi.\end{cases}$$

Therefore, the desired solution is

$$x(t)=\int_0^\pi G(t,s)f(s)ds$$

$$=\int_0^t\begin{pmatrix}\cos t\cos s & -\cos t\sin s\\ -\sin t\cos s & \sin t\sin s\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}ds-\int_t^\pi\begin{pmatrix}\sin t\sin s & \sin t\cos s\\ \cos t\sin s & \cos t\cos s\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}ds$$

$$=\int_0^t\begin{pmatrix}-\cos t\sin s\\ \sin t\sin s\end{pmatrix}ds-\int_t^\pi\begin{pmatrix}\sin t\cos s\\ \cos t\cos s\end{pmatrix}ds$$

$$=\begin{pmatrix}\cos t(\cos t-1)\\ -\sin t(\cos t-1)\end{pmatrix}+\begin{pmatrix}\sin^2 t\\ \sin t\cos t\end{pmatrix}=\begin{pmatrix}1-\cos t\\ \sin t\end{pmatrix}.$$

# Homework

1. Find the solution of the following BVP:

$$\begin{cases} x_1' = x_2 + t, & x_1(0) = 0 \\ x_2' = 1, & x_1(1) = 0 \end{cases},$$

where $\Phi(t) = e^{At} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ is given. (Answer: $\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} t^2 - t \\ t - 1 \end{pmatrix}$)


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